WEEK 3: PLAYING GAMES

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These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Games

Two player games are very common. A *winning strategy* for one of the two players (Alice) is a set of rules to follow, such that no matter what the other player does, if Alice follows the rules she will win the game. It is a fact that if Alice and Bob play a game which ends in a finite amount of time, and one of the two players always wins, then there is a winning strategy for either Alice or Bob. This is by no means a trivial remark!

Most games we play are too complicated to be solved (or else they wouldn't be much fun!) but even so, they contain little subgames which can - and often are - analyzed completely. For example, think of end games in chess.Here is an example of a simple game and a winning strategy for it:

- (The matchstick game) Alice and Bob play a game with a pile of 10 matches, with Alice moving first. On each players turn, they must remove between 1 and 3 matches from the pile. The person who empties the pile wins.The initial pile has 10 matches. Figure out a winning strategy for one of the players.
- (Misère matchsticks)Alice and Bob play the same game as above, only now the player who empties the pile *loses*. Figure out a winning strategy for one of the players.
- A polyhedron has at least 5 faces, and it has exactly 3 edges at each vertex. Two players play a game. Each in turn selects a face not previously selected. The winner is the first to get three faces with a common vertex. Show that the first player can always win.

- Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)
- The game of Chomp is played on an $m \times n$ board by Alice and Bob as follows. Alice moves first. On a players move, they must place an X on any square (i, j) which does not yet have an X on it, and they also place an X on any square above and to the right of that square which does not yet have an X on it. That is, any square (s, t) with $s \geq i$ and $t \geq j$ which does not yet have an X also gets an X put in it. The person who places the last X loses. Determine a winning strategy on a 3×3 game of chomp. What about on a 100×100 game of chomp? Now figure out who wins on a 100×101 game of chomp *hint, it is unknown what a winning strategy is!*

Here are some problems to get you started:

(1) Alice and Bob again play the matchstick game only now there are 100 matches, and the players may remove 2^m matches for any non-negative integer m (so you may remove $1, 2, 4, 8, \dots$ matches). Figure out a winning strategy for one of the players (last player to empty the pile wins).
(2) (harder) Alice and Bob play the matchstick game again, but now there are n matches in the pile, and they are allowed to remove $1, 3$ or 8 matches at a time. Alice plays first and the person to empty the pile wins. For which n does Alice have a winning strategy?
(3) Alice and Bob play a game in which the first player places a king on an empty 88 chessboard, and then, starting with the second player, they alternate moving the king (in accord with the rules of chess) to a square that has not been previously occupied. The player who moves last wins. Which player has a winning strategy? What about on a 5×5 board?
(4) Two players play a game on a 3×3 board. The first player places a 1 on an empty square and the second player places a 0 on an empty square. Play continues until all squares are occupied. The second player wins if the resulting determinant is 0 and the first player wins if it has any other value. Who wins?
(5) Alice and Bob play a game as follows. They start with a row of 50 coins, of various values. The players alternate, and at each step they pick either the first or last coin and take it. If Alice plays first, prove that she can guarantee that she will end up with at least as much

money as Bob. Find an example where Bob can make more money than alice if there are 51 coins.

(6) Same question(s) as above, only now with a knight instead of a king.

(7) Alice and Bob alternately draw diagonals between vertices of a regular polygon. They may connect two vertices if they are non-adjacent (i.e. not a side) and if the diagonal formed does not cross any of the previous diagonals formed. The last player to draw a diagonal wins. Who has a winning strategy if the polygon has 90 sides and Alice moves first? *Hint: START WITH SMALL CASES!*

(8) Let n be a positive integer. Alice and Bob play a game with a set of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of the players. Beginning with Alice, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$, and the last player to discard wins the game. Prove that Bob has a winning strategy.

(9) Bob is presented with two distinct positive integers , each concealed in its own envelope. Bob chooses one of the envelopes according to a flip of his trusty fair coin. He reveals the number contained in the envelope and guesses whether the number in the other envelope is larger or smaller than the number he just revealed. Does Bob have a strategy that gives him a better than even chance of guessing correctly? *not really a game theory problem, but too fun for me to resist!*